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Duality and the Renormalization Group*

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Abstract

The requirement that duality and renormalization group transformations commute as motions in the space of a theory has recently been explored to extract information about the renormalization flows in different statistical and field theoretical systems. After a review of what has been accomplished in the context of $2d$ sigma models, new results are presented which set up the stage for a fully generic calculation at two-loop order, with particular emphasis on the question of scheme dependence.

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1. Introduction

Duality symmetries are typically transformations in the parameter space of a theory which leave the partition function and the correlators invariant (perhaps up to some known function of the parameters). Renormalization group (RG) transformations also act in this same space, with similar invariance properties. Given the generality of this observation, one expects that it may be possible to investigate the interplay between duality and the RG whenever a system presents a duality symmetry and a renormalization flow, regardless of whether it is a quantum field theory, a statistical system, a lattice theory, etc..

Indeed, such a nontrivial interplay has recently been verified in a number of different contexts [1]-[5] and a requirement of consistency of duality symmetry and the RG has been used to obtain constraints on the RG flows. Spin systems, which generally enjoy a Kramers-Wannier duality symmetry, were considered in [1]. The Quantum Hall system, on the other hand, is strongly suspected to exhibit a much richer duality, under $SL(2,\mathbb{Z})$ or one of its (level 2) subgroups, and it was studied in [2]. For the spin systems considered in [1], the parameter space consisted of a single relevant coupling (the inverse temperature), while Kramers-Wannier duality forms the (colloquially speaking) simple group $\mathbb{Z}_2$, so that the constraints on the RG structure end up not being strong enough to actually determine unique flows. A more restrictive scenario appears in the Quantum Hall system, where the parameter space consists of the upper complex half-plane, and the requirements of holomorphy and modular symmetry naturally turn out to be considerably richer. Yet, even in that case, the existing results are not entirely conclusive: while on the one hand there is not enough experimental data confirming the precise symmetry group of the system, on the other hand by postulating a specific modular symmetry one still does not obtain unique RG flows.

Two-dimensional sigma models targeted on an arbitrary background of metric, antisymmetric tensor and dilaton fields also present a duality symmetry (when the background has an abelian isometry), and in that context, the situation is more favorable: while the symmetry group is, like for spin systems, also $\mathbb{Z}_2$, the parameter space is of course much larger, and the action of the group on it rather more involved, with geometry and torsion mixing in a nontrivial way. For the loop orders and backgrounds considered so far, this has in fact allowed for an essentially complete determination of the RG flows using only the requirement of consistency between duality and the RG (the qualification 'essentially' will be understood more clearly below). It is to these models and the relevant consistency requirements that the bulk of what follows will be dedicated.

To begin, we consider a system with a number of couplings, $k^i, i = 1, \ldots n$, and a duality symmetry, T, such that

$$Tk^i \equiv \tilde{k}^i = \tilde{k}^i(k) \tag{1.1}$$

represents a map between equivalent points in the parameter space (with equivalence taking the same meaning as, for instance, the order-disorder equivalence in the $2d$ Ising model). We will also assume the system has a renormalization group flow, R, encoded by a set of beta functions:

$$Rk^i \equiv \beta^i(k) = \mu \frac{d}{d\mu} k^i \ , \tag{1.2}$$

with μ some appropriate subtraction scale. On a generic function in the parameter space, $F(k)$, these operations act as follows:

$$\begin{aligned} TF(k) = &\ F(\tilde{k}(k)) \\ RF(k) = &\ \frac{\delta F(k)}{\delta k^i} \cdot \beta^i(k) \ . \end{aligned} \tag{1.3}$$

For a finite number of couplings the derivatives above should be understood as ordinary derivatives, whereas in the case of the sigma model these will be functional derivatives, and the dot will imply an integration over spacetime. The consistency requirement governing the interplay of duality and the RG can now be stated very simply:

$$[T, R] = 0 \tag{1.4}$$

or, in words, that duality transformations and RG flows commute as motions in the parameter space of the theory. This is the main concept to be explored, and from which most results will follow. It is easy to see that the above amounts to the following consistency conditions:

$$\beta^i(\tilde{k}) = \frac{\delta \tilde{k}^i}{\delta k^j} \cdot \beta^j(k) \ , \tag{1.5}$$

that is, under duality transformations the beta function must transform as a "form-invariant contravariant vector" (to avoid confusion: we are borrowing the language of General Relativity here, but of course duality transformations have nothing to do with diffeomorphisms!). It is this "form-invariance", *i.e.*, the fact that the functional form of β^i on the l.h.s. above must be the same as the one on the r.h.s., that is mostly responsible for the severity of the constraints engendered.

For the $2d$ Ising model on a square lattice this yields a constraint which is nontrivially satisfied by the (known) beta function of the model, although it does not determine uniquely this beta function. In the Quantum Hall system, on the other hand, the resulting constraint is that the beta function transform as a weight -2 modular form (strictly speaking, negative weight modular forms do not exist, and this obstruction is then circumvented by slightly relaxing the condition of holomorphy, but these details will not concern us here).

In what follows, we will explore in detail the analogous constraints in the context of $2d$ bosonic sigma models, in order of increasing complexity: Sections 2 and 3 contain a

review of previously published work [3],[4],[5] on, respectively, the fully generic one-loop case and the purely metric two-loop case, while Section 4 comprises results obtained in the course of more recent investigations [6], and presents the setup for the fully generic two-loop case, in the presence of torsion. In this case, where all possible backgrounds are included, the issue of scheme dependence will also be discussed in some detail, as it arises for the first time to complicate matters in a considerable way.

2. Sigma Models at One-Loop Order

Our starting point is the $d=2$ bosonic sigma model on a generic $D+1$-dimensional background $\{g_{\mu\nu}(X), b_{\mu\nu}(X)\}$ of metric and antisymmetric tensor, respectively, where $\mu,\nu = 0,1,\ldots,D = 0,i$, so that the $\mu=0$ component is singled out. We shall assume this sigma model has an abelian isometry, which will enable duality transformations, and we shall consider the background above in the adapted coordinates, in which the abelian isometry is made manifest through independence of the background on the coordinate $\theta \equiv X^0$. The original sigma model action reads:

$$
\begin{aligned}
S = \frac{1}{4\pi\alpha'} \int d^2\sigma \; \big[& g_{00}(X)\partial_\alpha\theta\partial^\alpha\theta + 2g_{0i}(X)\partial_\alpha\theta\partial^\alpha X^i + g_{ij}(X)\partial_\alpha X^i\partial^\alpha X^j \\
& + i\varepsilon^{\alpha\beta}\left(2b_{0i}(X)\partial_\alpha\theta\partial_\beta X^i + b_{ij}(X)\partial_\alpha X^i\partial_\beta X^j\right)\big] \; .
\end{aligned}
\tag{2.1}
$$

Throughout, all background tensors can depend only on target coordinates X^i, $i = 1,\ldots,D$, and not on θ.

The duality transformations in this model are also well-known [7]:

$$
\begin{aligned}
&\tilde{g}_{00} = \frac{1}{g_{00}} \quad , \quad \tilde{g}_{0i} = \frac{b_{0i}}{g_{00}} \quad , \quad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}} \; , \\
&\tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}} \quad , \quad \tilde{b}_{ij} = b_{ij} - \frac{g_{0i}b_{0j} - b_{0i}g_{0j}}{g_{00}} \; .
\end{aligned}
\tag{2.2}
$$

The statement of classical duality is that the model defined on the dual background $\{\tilde{g}_{\mu\nu}, \tilde{b}_{\mu\nu}\}$ is simply a different parametrization of the same model, given that the manipulations used to derive the transformations essentially only involve performing trivial integrations in a different order starting from the path-integral in which the abelian isometry is gauged.

On a curved worldsheet, another background must be introduced, that of the dilaton $\phi(X)$, coupling to the worldsheet curvature scalar. The RG flow of background couplings is given by their respective beta functions:

$$
\beta^g_{\mu\nu} \equiv \mu\frac{d}{d\mu}g_{\mu\nu} \; , \; \beta^b_{\mu\nu} \equiv \mu\frac{d}{d\mu}b_{\mu\nu} \; , \; \beta^\phi \equiv \mu\frac{d}{d\mu}\phi \; , \tag{2.3}
$$

while the trace of the stress energy tensor is found from the Weyl anomaly coefficients [8]

$$
\begin{aligned}
\bar{\beta}^g_{\mu\nu} =&\beta^g_{\mu\nu} + 2\alpha' \nabla_\mu \partial_\nu \phi + \nabla_{(\mu} W_{\nu)} \ , \\
\bar{\beta}^b_{\mu\nu} =&\beta^b_{\mu\nu} + \alpha' H_{\mu\nu}{}^\lambda \partial_\lambda \phi + H_{\mu\nu}{}^\lambda W_\lambda + \nabla_{[\mu} L_{\nu]} \ , \\
\bar{\beta}^\phi =&\beta^\phi + \alpha' (\partial_\mu \phi)^2 + \nabla^\mu \phi W_\mu \ ,
\end{aligned}
\tag{2.4}
$$

where W_μ and L_μ are some specific target vectors depending on $g_{\mu\nu}$ and $b_{\mu\nu}$, and $(\mu\nu) = \mu\nu + \nu\mu, [\mu\nu] = \mu\nu - \nu\mu$. For the loop orders and backgrounds considered in Sections 2 and 3, $W_\mu = L_\mu = 0$, and we will henceforth disregard them. Both the beta functions and the Weyl anomaly coefficients turn out to satisfy the consistency conditions, (1.5). However, while the latter satisfy them exactly, the former satisfy them up to a target reparametrization [3],[4]. Since both encode essentially the same RG information, for simplicity we will consider RG motions as generated by the Weyl anomaly coefficients in what follows. Thus, in the present context, the couplings are denoted by $k^i = \{g_{\mu\nu}, \beta_{\mu\nu}, \phi\}$, with $i = g, b, \phi$ labeling metric, antisymmetric tensor and dilaton backgrounds, and our R operation will in this case be defined, on a generic functional $F[g, b, \phi]$ (and in principle at any loop order), to be

$$
RF[g,b,\phi] = \frac{\delta F}{\delta g_{\mu\nu}} \cdot \bar{\beta}^g_{\mu\nu} + \frac{\delta F}{\delta b_{\mu\nu}} \cdot \bar{\beta}^b_{\mu\nu} + \frac{\delta F}{\delta \phi} \cdot \bar{\beta}^\phi \ , \tag{2.5}
$$

where the dot also indicates a spacetime integration. Duality transformations are given by

$$
TF[g,b,\phi] = F[\tilde{g}, \tilde{b}, \tilde{\phi}] \ , \tag{2.6}
$$

where $\tilde{\phi}$ will be defined shortly.

The consistency conditions to be satisfied, (1.5), that obtain from (2.2) translate more explicitly into:

$$
\begin{aligned}
\bar{\beta}^{\tilde{g}}_{00} =& -\frac{1}{g_{00}^2} \bar{\beta}^g_{00} \ , \\
\bar{\beta}^{\tilde{g}}_{0i} =& -\frac{1}{g_{00}^2} \left(b_{0i} \bar{\beta}^g_{00} - \bar{\beta}^b_{0i} g_{00} \right) \ , \\
\bar{\beta}^{\tilde{b}}_{0i} =& -\frac{1}{g_{00}^2} \left(g_{0i} \bar{\beta}^g_{00} - \bar{\beta}^g_{0i} g_{00} \right) \ , \\
\bar{\beta}^{\tilde{g}}_{ij} =& \bar{\beta}^g_{ij} - \frac{1}{g_{00}} \left(\bar{\beta}^g_{0i} g_{0j} + \bar{\beta}^g_{0j} g_{0i} - \bar{\beta}^b_{0i} b_{0j} - \bar{\beta}^b_{0j} b_{0i} \right) \\
&+ \frac{1}{g_{00}^2} \left(g_{0i} g_{0j} - b_{0i} b_{0j} \right) \bar{\beta}^g_{00} \ , \\
\bar{\beta}^{\tilde{b}}_{ij} =& \bar{\beta}^b_{ij} - \frac{1}{g_{00}} \left(\bar{\beta}^g_{0i} b_{0j} + \bar{\beta}^b_{0j} g_{0i} - \bar{\beta}^g_{0j} b_{0i} - \bar{\beta}^b_{0i} g_{0j} \right) \\
&+ \frac{1}{g_{00}^2} \left(g_{0i} b_{0j} - b_{0i} g_{0j} \right) \bar{\beta}^g_{00} \ ,
\end{aligned}
\tag{2.7}
$$

where, in a condensed notation, we take the quantities on the l.h.s. above to mean $\bar\beta^{\tilde g}_{\mu\nu} \equiv \bar\beta^{g}_{\mu\nu}[\tilde g, \tilde b, \tilde\phi]$, etc.. Both the dilaton duality transformation and its attendant consistency condition are still ostensibly missing, but will be determined shortly.

At loop order ℓ, the possible tensor structures $T_{\mu\nu}$ appearing in the beta function must scale as $T_{\mu\nu}(\Lambda g, \Lambda b) = \Lambda^{1-\ell} T_{\mu\nu}(g, b)$ under global scalings of the background fields. At $\mathcal{O}(\alpha')$ one may then have

$$\begin{aligned} \beta^g_{\mu\nu} =& \alpha' \left(A\, R_{\mu\nu} + B\, H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + C\, g_{\mu\nu} R + D\, g_{\mu\nu} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \right) \ , \\ \beta^b_{\mu\nu} =& \alpha' \left(E\, \nabla^\lambda H_{\mu\nu\lambda} \right) \ , \end{aligned} \tag{2.8}$$

where $H_{\mu\nu\lambda} = \partial_\mu b_{\nu\lambda} +$ cyclic permutations and with A, B, C, D, E being determined from one-loop Feynman diagrams. As found in [3], requiring (2.7) to be satisfied, and choosing $A = 1$ determines $B = -1/4$, $E = -1/2$, and $C = D = 0$, independently of any diagram calculations. The consistency conditions, (2.7), on $g_{\mu\nu}$ and $b_{\mu\nu}$ alone also allow for an independent determination of the dilaton transformation (or "shift") $\tilde\phi = \phi - \frac{1}{2} \ln g_{00}$. From this shift, and (1.5), one obtains yet another consistency condition,

$$\bar\beta^{\tilde\phi} = \bar\beta^{\phi} - \frac{1}{2 g_{00}} \bar\beta^g_{00} \ , \tag{2.9}$$

from which one can finally find the dilaton beta function, thus completely determining all beta functions at this order:

$$\begin{aligned} \beta^g_{\mu\nu} =& \alpha' \left(R_{\mu\nu} - \frac{1}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} \right) \\ \beta^b_{\mu\nu} =& -\frac{\alpha'}{2} \nabla_\lambda H^\lambda{}_{\mu\nu} \\ \beta^\phi =& C - \frac{\alpha'}{2} \nabla^2 \phi \ - \frac{\alpha'}{24}\ H_{\mu\nu\lambda} H^{\mu\nu\lambda} \ , \end{aligned} \tag{2.10}$$

up to the constant C.

3. Two-Loop Order with Purely Metric Backgrounds

At the next order R is modified by the two-loop beta functions, and one must determine the appropriate modifications in T such that $[T, R] = 0$ continues to hold. We begin by working with restricted backgrounds of the form

$$g_{\mu\nu} = \begin{pmatrix} a & 0 \\ 0 & \bar g_{ij} \end{pmatrix} \ , \tag{3.1}$$

and $b_{\mu\nu} = 0$, so that no torsion appears in the dual background either. It is useful to define at this point the following two quantities: $a_i \equiv \partial_i \ln a$, and $q_{ij} \equiv \bar\nabla_i a_j + \frac{1}{2} a_i a_j$, where barred

quantities here and below refer to the metric $\bar{g}_{ij}$ (also, indices $i, j, \ldots$, are contracted with the metric $\bar{g}_{ij}$). Within this class of backgrounds classical duality transformations reduce to the operation $a \to 1/a$, and it is simple to determine the possible corrections to T from a few basic requirements, spelled out in detail in [5]. For conciseness, we will directly present the final result for the corrected duality transformations [9]:

$$\begin{aligned} \ln \tilde{a} =& -\ln a + \lambda\alpha' a_i a^i \ , \\ \tilde{g}_{ij} =& g_{ij} = \bar{g}_{ij} \ , \\ \tilde{\phi} =& \phi - \frac{1}{2}\ln a + \frac{\lambda}{4}\alpha' a_i a^i \ , \end{aligned} \tag{3.2}$$

where λ is a constant that cannot be determined from the basic requirements. The consistency conditions that follow from the above are:

$$\begin{aligned} \frac{1}{\tilde{a}}\ \bar{\beta}^{\tilde{g}}_{00} =& -\frac{1}{a}\bar{\beta}^{g}_{00} + 2\lambda\alpha' \left[a^i\partial_i\left(\frac{1}{a}\bar{\beta}^g_{00}\right) - \frac{1}{2}a^i a^j \bar{\beta}^g_{ij}\right] \ , \\ \bar{\beta}^{\tilde{g}}_{ij} =& \bar{\beta}^{g}_{ij} \ , \\ \bar{\beta}^{\tilde{\phi}} =& \bar{\beta}^{\phi} - \frac{1}{2a}\bar{\beta}^g_{00} + \frac{\lambda}{2}\alpha' \left[a^i\partial_i\left(\frac{1}{a}\bar{\beta}^g_{00}\right) - \frac{1}{2}a^i a^j \bar{\beta}^g_{ij}\right] \ . \end{aligned} \tag{3.3}$$

The terms scaling correctly under $g \to \Lambda g$ at this order, and thus possibly present in the beta function, are

$$\begin{aligned} \beta^{g(2)}_{\mu\nu} =& A_1\, \nabla_\mu\nabla_\nu R + A_2\, \nabla^2 R_{\mu\nu} + A_3\, R_{\mu\alpha\nu\beta}R^{\alpha\beta} + A_4\, R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + A_5\, R_{\mu\alpha}R_\nu{}^\alpha \\ & + A_6\, R_{\mu\nu}R + A_7\, g_{\mu\nu}\nabla^2 R + A_8\, g_{\mu\nu}R^2 + A_9\, g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + A_{10}\, g_{\mu\nu}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \end{aligned} \tag{3.4}$$

(we have used Bianchi identities to reduce from a larger set of tensor structures).

It will suffice in fact to study the consistency conditions for the (ij) components, $\bar{\beta}^{\tilde{g}}_{ij} = \bar{\beta}^{g}_{ij}$, in order to determine the only structure satisfying all the consistency conditions.

We write

$$\bar{\beta}^g_{ij} = \alpha'\left(\beta^{g(1)}_{ij} + 2\bar{\nabla}_i\partial_j\phi\right) + \alpha'^2\beta^{g(2)}_{ij} \ , \tag{3.5}$$

where $\beta^{g(1)}_{ij} = R_{ij} = \bar{R}_{ij} - \frac{1}{2}q_{ij}$ is the one-loop beta function, and perform the duality transformation (3.2), keeping terms to $\mathcal{O}(\alpha'^2)$. Using the fact that the one-loop Weyl anomaly coefficient satisfies the one-loop consistency conditions (2.7), we arrive at

$$\beta^{\tilde{g}(2)}_{ij} = \beta^{g(2)}_{ij} - \frac{1}{4}\lambda a_{(i}\partial_{j)}(a^k a_k) \ , \tag{3.6}$$

where the duality transformation of $\beta^{g(2)}_{ij}$ is given simply by $a \to 1/a$ without α' corrections, since this is already $\mathcal{O}(\alpha'^2)$. Separating the possible tensor structures into even and odd tensors under $a \to 1/a$,

$$\beta^{g(2)}_{ij} = E_{ij} + O_{ij} \ , \quad \tilde{E}_{ij} = E_{ij} \ , \quad \tilde{O}_{ij} = -O_{ij} \ , \tag{3.7}$$

the even structures drop out of (3.6) and we are left with

$$O_{ij} = \frac{1}{8}\lambda a_{(i}\partial_{j)}(a^k a_k) \ . \tag{3.8}$$

We now perform a standard Kaluza-Klein reduction on the ten terms in (3.4) to identify which if any satisfy this condition. The results can be obtained using the formulas in the Appendix of [4], and are as follows:

$$\begin{aligned}
(1):\ & \nabla_i\nabla_j R = \bar{\nabla}_i\bar{\nabla}_j(\bar{R} - q_n{}^n) \ , \\
(2):\ & \nabla^2 R_{ij} = (\bar{\nabla}^2 + \frac{1}{2}a_k\bar{\nabla}^k)(\bar{R}_{ij} - \frac{1}{2}q_{ij}) - \frac{1}{4}a_i a_j q_n{}^n \\
& \qquad - \frac{1}{4}a^k a_{(i}\left(\bar{R}_{j)k} - \frac{1}{2}q_{j)k}\right) \ , \\
(3):\ & R_{i\alpha j\beta}R^{\alpha\beta} = \frac{1}{4}q_{ij}q_n{}^n + \bar{R}_{injm}(\bar{R}^{nm} - \frac{1}{2}q^{nm}) \ , \\
(4):\ & R_{i\alpha\beta\gamma}R_j{}^{\alpha\beta\gamma} = \frac{1}{2}q_{ik}q_j{}^k + \bar{R}_{iknm}\bar{R}_j{}^{knm} \ , \\
(5):\ & R_{i\alpha}R_j{}^\alpha = \bar{R}_{ik}\bar{R}_j{}^k - \frac{1}{2}\bar{R}_{k(i}q_{j)}{}^k + \frac{1}{4}q_{ik}q_j{}^k \ , \\
(6):\ & R_{ij}R = (\bar{R}_{ij} - \frac{1}{2}q_{ij})(\bar{R} - q_n{}^n) \ , \\
(7):\ & g_{ij}\nabla^2 R = \bar{g}_{ij}\left[\frac{1}{2}a^k\partial_k(\bar{R} - q_m{}^m)\right. \\
& \qquad \left. + \bar{\nabla}^k\partial_k(\bar{R} - q_m{}^m)\right] \ , \\
(8):\ & g_{ij}R^2 = \bar{g}_{ij}\left(\bar{R} - q_m{}^m\right)^2 \ , \\
(9):\ & g_{ij}R_{\alpha\beta}R^{\alpha\beta} = \bar{g}_{ij}\left[\frac{1}{4}(q_m{}^m)^2 + (\bar{R}_{km} - \frac{1}{2}q_{km})^2\right] \ , \\
(10):\ & g_{ij}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \bar{g}_{ij}\left[q_{km}q^{km} + \bar{R}_{k\ell mn}\bar{R}^{k\ell mn}\right] \ .
\end{aligned} \tag{3.9}$$

The respective odd parts are

$$\begin{aligned}
O_{ij}^{(1)} =& -\bar{\nabla}_i\bar{\nabla}_j\bar{\nabla}_n a^n \ , \\
O_{ij}^{(2)} =& \frac{1}{2}a_k\bar{\nabla}^k\bar{R}_{ij} - \frac{1}{2}\bar{\nabla}^2\bar{\nabla}_i a_j - \frac{1}{4}a_i a_j\bar{\nabla}_k a^k \ , \\
O_{ij}^{(3)} =& -\frac{1}{2}\bar{R}_{injm}\bar{\nabla}^n a^m + \frac{1}{8}a_n a^n\bar{\nabla}_i a_j + \frac{1}{8}a_i a_j\bar{\nabla}_n a^n \ , \\
O_{ij}^{(4)} =& \frac{1}{4}a_k a_{(i}\bar{\nabla}_{j)}a^k \ ,
\end{aligned}$$

$$
\begin{aligned}
O^{(5)}_{ij} =& -\frac{1}{2}\bar{R}_{k(i}\bar{\nabla}_{j)}a^k + \frac{1}{8}a_k a_{(i}\bar{\nabla}_{j)}a^k \ , \\
O^{(6)}_{ij} =& -\frac{1}{2}\bar{R}\bar{\nabla}_i a_j - \bar{R}_{ij}\bar{\nabla}_n a^n + \frac{1}{4}a_i a_j \bar{\nabla}_n a^n \\
&+ \frac{1}{4}a_n a^n \bar{\nabla}_i a_j \ , \\
O^{(7)}_{ij} =& \bar{g}_{ij}\left[\frac{1}{2}a^k\partial_k(\bar{R} - \frac{1}{2}a_m a^m) - \bar{\nabla}^k\partial_k(\bar{\nabla}_m a^m)\right] \ , \\
O^{(8)}_{ij} =& \bar{g}_{ij}\left[-2(\bar{\nabla}^k a_k)\bar{R} + (\bar{\nabla}^k a_k)a^m a_m\right] \ , \\
O^{(9)}_{ij} =& \bar{g}_{ij}\left[\frac{1}{4}(\bar{\nabla}^k a_k)a^m a_m - (\bar{\nabla}_k a_m)\bar{R}^{km}\right. \\
&\left. + \frac{1}{4}(\bar{\nabla}_k a_m)a^k a^m\right] \ , \\
O^{(10)}_{ij} =& \bar{g}_{ij}(\bar{\nabla}_k a_m)a^k a^m \ .
\end{aligned}
\tag{3.10}
$$

It is fortunate that none of these tensors contain purely even structures, since such structures are left unconstrained (and thus undetermined) by duality. The only odd term of the form (3.8) comes from $A_4 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}$, and a detailed inspection shows that no linear combination of the other terms gives rise to odd tensors generically of the form (3.8). This determines that, with the requirement of covariance of duality under the RG, the $\mathcal{O}(\alpha'^2)$ term in the beta function is

$$
\beta^{g(2)}_{\mu\nu} = \lambda R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} \ . \tag{3.11}
$$

One should now check that the corresponding (00) component also satisfies its consistency condition. A straightforward computation shows that it does, and the determination of the two-loop beta function is thus complete.

Although we treated a restricted class of metric backgrounds, the final result is valid for a generic metric, since none of the possible tensor structures are built out of the off-block-diagonal g_{0i} elements alone (in which case our consistency conditions would be blind to them, just as they are to the even terms E_{ij}).

Simply using the requirements that duality and the RG commute as motions in the parameter space of the sigma model, we have thus been able to determine the two-loop beta function to be

$$
\beta_{\mu\nu} = \alpha' R_{\mu\nu} + {\alpha'}^2 \lambda R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} \ , \tag{3.12}
$$

for an entirely generic metric background, again without any Feynman diagram calculations. Because we used an extremely restrictive class of backgrounds, it was not possible to determine the value of λ (the correct value is $\lambda = \frac{1}{2}$). However, we expect that, similarly to what happens at $\mathcal{O}(\alpha')$, once a more generic background is used in the consistency

conditions, even this constant should be determined. We now examine how to go about calculating in such a generic background in an efficient way.

4. Setup for the Fully Generic Two-Loop Case

The inclusion of torsion at two-loop order brings with it a number of complications which one should try to minimize as much as possible. First, the number of new terms appearing in the beta functions is greatly increased as compared to the purely metric case. Moreover, there is now one more beta function to worry about, for the antisymmetric tensor. Also, it is clear that there will be several new terms in the corrections to duality transformations, and the general arguments used in [5] will not be sufficient to determine them. To finally complicate the situation further, the scheme dependence present leaves a lot more latitude to what the correct expressions for these beta functions are, as well as which duality transformations should make them transform covariantly.

It thus seems that a direct guessing of the corrected duality transformations, attempting to keep the two-loop beta functions covariant, would be an extremely arduous task, and we will try rather to first streamline the calculations involved by going through what may seem at first a longer path.

We start with the observation that there is a connection between the Weyl anomaly coefficients and the string background effective action ("EA" in what follows), whereby one establishes a direct relation between the former and the equations of motion of the latter. In principle, there is thus the possibility that the duality transformation properties of one imply the transformation properties of the other. Should this be the case, one might save considerable effort by studying the effective action alone, since this is a scalar function on the parameter space, invariant under duality, whereas the beta function represents a vector field in that space, with nontrivial transformation properties.

Unfortunately, we will see that the transformation properties of the EA under duality will *not* allow us to deduce the transformation properties of the Weyl anomaly coefficients. However, the detailed consideration of this relationship will still be useful, on the one hand to limit the possible transformations under which the Weyl anomaly coefficients behave covariantly, and on the other, to clarify the messy issue of scheme dependence.

We begin by examining the situation at one-loop order. The EA is given by

$$S \equiv \alpha' S_0 = \alpha' \int d\theta \; d^D x \; \sqrt{g} e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) \; . \qquad (4.1)$$

Given the one-loop expressions for the beta functions, (2.10), it is simple to see that this EA is actually equivalent to (with $C = 0$)

$$S = R\, V = V_{,i} \cdot \bar{\beta}^i \; , \qquad (4.2)$$

where $V = \sqrt{g}\exp -2\phi$, and $V_{,i} \equiv \delta V/\delta k^i$, in the notation of the introduction. Because we know the one-loop Weyl anomaly coefficients transform contravariantly under duality (cf. (1.5)), the gradient $\delta/\delta k^i$ transforms covariantly, and V is invariant, it immediately follows that S as defined above is invariant under duality transformations. Similarly, at higher loop orders, if we are able to find the corrected duality transformations under which the higher-loop Weyl anomaly coefficients transform contravariantly as in (1.5), and if we are able to find a scheme in which the EA continues to be given by (4.2), then we are guaranteed duality invariance of the EA. But that is actually opposite to the direction we are seeking. Can we attempt to argue also conversely? At first sight, (4.2) does seem to give this converse result, that once a duality transformation can be found at some loop order that keeps S invariant, that will imply the sought for contravariance of the Weyl anomaly coefficients, and thus the statement that $[T,R]=0$.

That is not correct, however, for $V_{,i}$, which is given more explicitly by

$$V_{,i} = \begin{pmatrix} V_{,g} \\ V_{,b} \\ V_{,\phi} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}Vg^{\mu\nu} \\ 0 \\ -2V \end{pmatrix} \ , \tag{4.3}$$

(we are omitting a delta function coming from the functional derivative) has an enormous amount of “zero modes”, given by

$$z^i = \begin{pmatrix} z^g \\ z^b \\ z^\phi \end{pmatrix} = \begin{pmatrix} z^g_{\mu\nu} \\ z^b_{\mu\nu} \\ \frac{1}{4}g^{\mu\nu}z^g_{\mu\nu} \end{pmatrix} \ , \tag{4.4}$$

with $z^g_{\mu\nu}, z^b_{\mu\nu}$ arbitrary, so that $V_{,i}\ z^i = 0$. This implies that, at some loop order, if there is a set of duality transformations that keeps S as defined in (4.2) invariant, then the Weyl anomaly coefficients are seen to transform not as in (1.5), but as

$$\beta^i(\tilde{k}) + \tilde{z}^i(\tilde{k}) = \frac{\delta\tilde{k}^i}{\delta k^j}\cdot\left(\beta^j(k) + z^j(k)\right) \ , \tag{4.5}$$

with $z^i(k)$ and $\tilde{z}^i(\tilde{k})$ some specific vectors of the form (4.4), not necessarily zero. Naturally, this does not represent any covariance property at all.

Could some other reasoning rescue the possibility that the invariance of the EA might imply the contravariance of the Weyl anomaly coefficients? For instance, one immediate criticism that may be applied to the argument above is that one is not sure *a priori* that the EA should really be given by (4.2). This brings us to the issue of scheme ambiguities.

An independent definition of the EA corresponds to the field theory action that generates the massless sector of the (string) tree level string S-matrix. That EA contains a large number of terms that are ambiguous in that they can be modified with field redefinitions of the EA, and it contains a smaller number of terms that are invariant under field

redefinitions, and thus unambiguous. Field redefinitions also affect the beta functions, and stemming from their definition, (2.3), it is simple to see that they must transform under field redefinitions as contravariant vectors (now we *are* talking about diffeomorphisms). Parenthetically, we note that a subset of these field redefinitions correspond to what is typically referred to as a change of subtraction scheme in the renormalization of the sigma model: if, say, minimal subtraction corresponds to the subtraction of a divergent term $1/\varepsilon\, T_{\mu\nu}$, then a different, nonminimal scheme corresponds to the subtraction of a term (const. $+ 1/\varepsilon$) $T_{\mu\nu}$, which in turn is equivalent to a field redefinition by the term (const.) $T_{\mu\nu}$. With such a notion of scheme ambiguity, it can be seen that the two-loop beta function in the purely metric case is actually scheme independent, a property which is lost when torsion is included. More generally, however, because the sigma model is not renormalizable in the usual sense, one is also allowed finite subtractions of terms not originally present in the action, and these correspond to arbitrary field redefinitions. In order not to propagate semantic confusion, we will refrain from using the standard (and more restrictive) notion of scheme ambiguity, and will always consider instead the full generality of arbitrary field redefinitions, referring to different redefinitions as different choices of scheme.

At any rate, we realize from the discussion above that there is an unambiguous and independent notion to the EA, and that to each different "realization" of it, or choice of scheme, there corresponds also a choice of scheme for the beta functions. It is expected that in any scheme there should be a relation between the equations of motion of the EA and the Weyl anomaly coefficients of the form

$$\frac{\delta S}{\delta k^i} = G_{ij} \cdot \bar{\beta}^j \ , \tag{4.6}$$

with G_{ij} invertible, in the sense that the equations of motion imply the vanishing of the Weyl anomaly coefficients, and vice-versa (an even stronger requirement would be the positivity of G_{ij}, in order to connect the EA to a c-function, but that will not concern us here). Could this relation between the EA and $\bar{\beta}^i$'s allow us to deduce the contravariance of the latter from invariance of the former?

The notation certainly is very suggestive, with G_{ij} naturally appearing to change an object transforming contravariantly into an object transforming covariantly. However, insofar as G_{ij} itself has no independent meaning,† but is *devised* to have the above equation satisfied (in the sense that it just represents the particular linear combinations of $\bar{\beta}^i$'s that

† Again, in the context of a c-theorem it would, but scheme dependence in the present context complicates matters too much to allow one at this point to seriously conjecture G_{ij} to be the Zamolodchikov metric. This may well turn out to be true eventually, in some scheme, but we shall simply not assume it here.

yield the equations of motion of the EA), the answer is again unfortunately negative, and it is well exemplified by the situation at one loop order already. With the beta functions given by (2.10), and the EA by (4.1), it is simple to find that G_{ij} will be

$$
\begin{aligned}
G_{ij} &= \begin{pmatrix} G_{gg}^{\mu\nu\lambda\rho} & G_{gb}^{\mu\nu\lambda\rho} & G_{g\phi}^{\mu\nu} \\ G_{bg}^{\mu\nu\lambda\rho} & G_{bb}^{\mu\nu\lambda\rho} & G_{b\phi}^{\mu\nu} \\ G_{\phi g}^{\lambda\rho} & G_{\phi b}^{\lambda\rho} & G_{\phi\phi} \end{pmatrix} \\
&= \sqrt{g}e^{-2\phi} \begin{pmatrix} \frac{1}{2}g^{\mu\nu}g^{\lambda\rho} - \frac{1}{2}g^{\mu(\lambda}g^{\rho)\nu} & 0 & -2g^{\mu\nu} \\ 0 & -\frac{1}{2}g^{\mu[\lambda}g^{\rho]\nu} & 0 \\ -2g^{\lambda\rho} & 0 & 8 \end{pmatrix} .
\end{aligned}
\tag{4.7}
$$

We already know that both the one-loop Weyl anomaly coefficients and EA transform in the proper way, and thus one *must* find, if one were to check explicitly the transformation properties of the particular G_{ij} given above, that it transforms like a rank 2 form-invariant covariant tensor under duality. If we did not know how the $\bar{\beta}^i$'s transformed, however, all we could tell from the invariance of S is that G_{ij} has to cancel whatever (possibly completely wrong) transformation property of $\bar{\beta}^j$, and yield the transformation rule for a covariant vector. Thus if, say, the antisymmetric tensor Weyl anomaly coefficient were twice its correct value, (4.6) would still hold if we multiplied G_{bb} by 1/2, and yet the "new" $\bar{\beta}^i$ (with the wrong coefficient of $\bar{\beta}^b$) would certainly *not* satisfy (1.5), and consequently $[T, R] = 0$ would also not be satisfied. Accordingly, the "new" G_{ij} would also not transform like a rank 2 covariant tensor. Another clear, and even more pertinent, example of this can be seen with G_{ij} at two-loop order: if (4.6) is expanded to $\mathcal{O}(\alpha'^2)$, the r.h.s. will contain, at $\mathcal{O}(\alpha'^2)$, a term given by the contraction of G_{ij} at two-loop order with $\bar{\beta}^j$ at one-loop order. Such an expansion is considered in [10], and the authors note there that because G_{ij} at two-loop order has the same tensor structures as $\bar{\beta}^j$ at one-loop order, whenever a term appears containing (roughly speaking) the square of a one-loop $\bar{\beta}^j$, it becomes impossible to determine which piece belongs to G_{ij}, and which to $\bar{\beta}^j$. Of course, any choice other than the correct one will lead to a violation of $[T, R] = 0$, even though (4.6) is perfectly satisfied whichever way these terms are split up. Incidentally, the authors of [10] suggest the only way to resolve this indefinition in G_{ij} is by going one order higher; we would suggest instead that the present considerations involving duality will eventually resolve this problem without the need to go to three-loop order.

So far, it has seemed that the invariance of the EA cannot really be of any help in determining the covariance properties of the Weyl anomaly coefficients. But, in fact, the above has not been entirely in vain: we know that, in the scheme in which the EA is given by (4.2) at higher loop order, if there exists at all any duality transformations that respect $[T, R] = 0$, *i.e.*, such that $\bar{\beta}^i$ transforms contravariantly, then these transformations must

keep the EA invariant; thus, if there is only one set of transformations that keep the EA invariant, these are the only transformations that have a chance of satisfying $[T, R] = 0$. So, we are not guaranteed that the transformations that keep S invariant satisfy $[T, R] = 0$, but if we know the only transformations that keep S invariant, we are at least not groping in the dark trying to guess which transformations we should be testing on the Weyl anomaly coefficients.

Recently, a set of corrected duality transformations has been found [11] that keep invariant the two-loop EA in a particular scheme. In that scheme, it is claimed that the matrix G_{ij} connecting the equations of motion and the Weyl anomaly coefficients is purely numerical, that is, it contains no spacetime derivative operators acting on $\bar{\beta}^i$ [10],[12]. That is certainly a crucial advantage if one is interested in studying a c-theorem for generalized sigma models. For our purposes, however, the disadvantage of that scheme is the fact that the expression for the EA is very complicated, containing a large number of scheme dependent terms as compared to the "minimal" EA that reproduces string scattering amplitudes. Furthermore, that EA does not satisfy (4.2), a property we would like to maintain; instead, the so-called "minimal" EA, S_{min}, does [13]. We would therefore like to obtain all our results in that scheme if possible.

In order to do this, we will study the general problem of scheme dependence, to determine whether we can find a set of duality transformations that keeps an EA invariant in one scheme if another set of transformations is given that keeps the EA invariant in another scheme.

In the generic notation of the introduction, we assume we are given an EA in one particular scheme to two-loop order,

$$S(k) = \alpha' S_0(k) + \alpha'^2 \ S_1(k) \ , \tag{4.8}$$

and a set of (two-loop corrected) duality transformations

$$\tilde{k}^i(k) = \tilde{k}^i_0(k) + \alpha' \ \tilde{k}^i_1(k) \ , \tag{4.9}$$

such that $S(\tilde{k}) = S(k)$ to $\mathcal{O}(\alpha'^2)$. Thus, $S_0(k)$ is given by (4.1), $S_1(k)$ may be for instance the two-loop EA in the scheme considered in [10],[11], $\tilde{k}_0(k)$ is given by (2.2), and $\tilde{k}_1(k)$ would then be the corrections to duality found in [11]. We now make a field redefinition

$$\hat{k}^i(k) = k^i + \alpha' \ f^i(k) \ , \tag{4.10}$$

with $f^i(k)$ some functional of the couplings with the appropriate dimensions. The field redefined EA, to $\mathcal{O}(\alpha'^2)$, will be

$$\hat{S}(k) \equiv S(\hat{k}) = S(k) + \alpha' \ f^i(k) \cdot \frac{\delta S(k)}{\delta k^i} \ . \tag{4.11}$$

To the order considered, the invariance $S(\tilde{k}) = S(k)$ assumed implies

$$\begin{aligned} S_0(k) =& S_0(\tilde{k}_0) \\ S_1(k) =& S_1(\tilde{k}_0) + \tilde{k}_1^i(k) \cdot \frac{\delta S_0(\tilde{k}_0)}{\delta \tilde{k}_0^i} \ . \end{aligned} \tag{4.12}$$

We now write

$$\tilde{\kappa}^i(k) = \tilde{k}_0^i(k) + \alpha' \ \tilde{\kappa}_1^i(k) \tag{4.13}$$

for the duality transformations that will keep the field redefined EA, $\hat{S}(k)$, invariant:

$$\hat{S}(\tilde{\kappa}(k)) = \hat{S}(k) \ . \tag{4.14}$$

To determine these field redefined duality transformations, one must now substitute (4.13) into (4.14), using (4.10),(4.11),(4.12), and keeping terms to $\mathcal{O}(\alpha'^2)$. This is done in a straightforward way, and we simply state the final result:

$$\tilde{\kappa}_1^i(k) = \tilde{k}_1^i(k) + \left(\frac{\delta \tilde{k}_0^i}{\delta k^j} \cdot f^j(k) - f^i(\tilde{k}_0) \right) \ . \tag{4.15}$$

This is the result we sought: given a set of transformations keeping the EA invariant in some scheme, we can explicitly construct the set of transformations keeping the EA invariant in any other scheme. It is interesting to note that the term in parenthesis on the r.h.s. above represents precisely the one-loop consistency conditions $[T, R] = 0$, but acting on the field redefinitions rather than on the Weyl anomaly coefficients. In other words, in changing from one scheme to another through a field redefinition, the duality transformations that keep the redefined EA invariant differ from the original transformations by a term which "corrects" for how much off the field redefinitions themselves are from satisfying the one-loop consistency conditions.

The minimal EA, $S_{min} = \alpha' S_0 + \alpha'^2 \ S_{1min}$,

$$\begin{aligned} S_{1min} =& \frac{1}{4} \int d\theta \ d^D x \ \sqrt{g} e^{-2\phi} \left(R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - \frac{1}{2} R^{\mu\nu\lambda\rho} H_{\mu\nu\sigma} H_{\lambda\rho}{}^{\sigma} \right. \\ & \left. + \frac{1}{24} H_{\mu\nu\lambda} H^{\nu}{}_{\rho\alpha} H^{\rho\sigma\lambda} H_{\sigma}{}^{\mu\alpha} - \frac{1}{8} H_{\mu\alpha\beta} H_{\nu}{}^{\alpha\beta} H^{\mu\rho\sigma} H^{\nu}{}_{\rho\sigma} \right) \ , \end{aligned} \tag{4.16}$$

the field redefinition taking the nonminimal action of [10],[11] into it, and the beta functions in several different subtraction schemes, can all be gleaned from the literature [10],[13],[14]. The task at hand is now to find the duality transformations that keep (4.16) invariant and, using those as the operation T, and the beta functions in the appropriate scheme to define R, verify whether $[T, R] = 0$ holds at two-loop order. It should be noted that what we have done above *guarantees* that there exists a set of duality transformations that keeps S_{min} invariant; however, the constructive procedure, in (4.15), of obtaining these transformations starting from the transformations found in [11], is very likely not the most efficient way to proceed, and we have opted instead for direct guessing and verification on S_{min}. We expect to report on this in the near future [6].

5. Conclusions

The requirement that duality and the RG commute as motions in the parameter space of a model is a very basic one, and it has been shown not only to be verified in the instances it has been tested, but also to yield important constraints on the RG flows in the context of $2d$ sigma models.

While at one-loop order this interplay between duality and the RG in the sigma model has been thoroughly investigated, at two-loop order the analysis has not been exhaustive so far. To help us in achieving this complete analysis, we have available first of all a set of duality transformations keeping a string background effective action invariant [11]. We have shown that there is no guarantee that the set of transformations that keeps this effective action invariant will also turn out to satisfy the duality consistency conditions $[T, R] = 0$. However, we have also seen that if any transformations at all exist that do satisfy the consistency conditions, they must also keep the effective action invariant (at least in the "minimal" scheme), so that by finding the transformations that keep the effective action invariant one is selecting the one set of transformations that has a chance of satisfying $[T, R] = 0$.

We believe this basic statement, $[T, R] = 0$, to be a more fundamental feature of the models in question than the invariance of the string background effective action, which follows from it (and which only is defined for sigma models). This has represented sufficient motivation for us to delve into the question of its validity in full generality at two-loop order, with the encouragement that the existence of a duality invariance of the string background effective action has already been shown in the same context.

Field redefinition ambiguities enter at this loop order as an added complication. We have taken the first step in comprehensively accounting for them by establishing that duality symmetry is a well-defined notion over and above the presence of such ambiguities, in the sense that if it is present in one choice of scheme, it may be modified but will nonetheless also be present in any other scheme.

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